# Mathematics - I Semester

Mathematics, I Semester, I year imp Questions

①

## Unit - I

① If $y = a\cos(\log x) + b\sin(\log x)$, show that
$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

② If $ax^2 + 2hxy + by^2 = 1$, show that $\frac{d^2y}{dx^2} = \frac{h^2 - ab}{(hx+by)^3}$

③ If $y = e^{-x}(A\cos x + B\sin x)$, prove that $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$

④ If $y = (\tan^{-1}x)^2$, prove that $(x^2+1)^2 \frac{d^2y}{dx^2} + 2x(x^2+1)\frac{dy}{dx} - 2 = 0$

⑤ Find the nth derivative of a) $y = \sin^{-1}\frac{2x}{1+x^2}$

b) $y = \tan^{-1}\frac{1+x}{1-x}$

⑥ Find the nth derivative of $y = e^{2x}\cos x \sin^2 2x$

⑦ If $y^{1/m} + y^{-1/m} = 2x$ prove that $(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$

⑧ If $y = \cos(m\sin^{-1}x)$, show that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0$

⑨ If $y = e^{a\sin^{-1}x}$ prove that $\cancel{(1+x^2)y_{n+1}} + \cancel{(2nx-1)}\,y_n \cancel{+n(}$

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0$$

⑩ prove that $e^x\cos x = 1 + x - \frac{2x^3}{3!} - \frac{2^2x^4}{4!} - \frac{2^2x^5}{5!} \cdots$

(By using Maclurin's theorem)

(11) Expand $e^{a\sin^{-1}x}$ in powers of $x$ by Maclaurin's theorem and hence obtain the value of $e^{\theta}$.

(12) Expand $\sin x$ in powers of $(x-\frac{\pi}{2})$

(13) use Taylor's theorem to prove that

$$\tan^{-1}(x+h) = \tan^{-1}x + h\sin z \cdot \frac{\sin z}{1} - (h\sin z)^2 \frac{\sin 2z}{2} + \cdots$$

where $z = \cot^{-1}x$.

(14) State and prove Rolle's theorem,

(15) State and prove Lagrange's meanvalue theorem

(16) state and prove cauchy's mean value theorem

(17) verify Rolle's theorem, for the function

i) $f(x) = \log[(x^2+ab)/(a+b)x]$ ; $x \in [a,b]$

ii) $f(x) = 2+(x-1)^{2/3}$, $x \in [0,2]$

(18) Verify the mean value theorem for

i) $f(x) = \log x$ in $[1,e]$

ii) $f(x) = x^3$ in $[a,b]$

iii) $f(x) = lx^2+mx+n$ in $[a,b]$

(19) show that $\frac{v-u}{1+v^2} < \tan^{-1}v - \tan^{-1}u < \frac{v-u}{1+v^2}$, $0<u<v$.

and deduce that $\frac{\pi}{4}+\frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4}+\frac{1}{6}$

## Unit - II



(1) Determine $\lim \dfrac{e^x - e^{-x} - 2\log(1+x)}{x\sin x}$, where $x \to 0$

(2) Evaluate the following

a) $\lim\limits_{x\to 0} \dfrac{x\cos x - \log(1+x)}{x^2}$ b) $\dfrac{\tan x - x}{x^2 \sin x}$ $(x \to 0)$

c) $\lim\limits_{x\to 0} \dfrac{xe^x - \log(1+x)}{x^2}$ d) $\dfrac{\sin x - x + \frac{x^3}{6}}{x^5}$ $(x \to 0)$

(3) Determine the following limits

a) $\left(\dfrac{1}{x} - \dfrac{1}{e^x - 1}\right)$ $(x \to 0)$ b) $\left(\dfrac{a}{x} - \cot\dfrac{x}{a}\right)$ $(x \to 0)$

c) $(\sec x - \tan x)$ $(x \to \frac{\pi}{2})$ d) $\left(\dfrac{1}{x} - \dfrac{1}{\sin x}\right)$ $(x \to 0)$

(4) Determine $\lim \left(2 - \dfrac{x}{a}\right)^{\tan\frac{\pi x}{2a}}$ as $x \to a$

(5) Determine $\lim (\sin x)^{\tan x}$ $(x \to \pi/2)$

(6) Find $\dfrac{ds}{dx}$ for the curves

(i) $y = \cosh\left(\dfrac{x}{c}\right)$ (2) $a\log\left[\dfrac{a^2}{a^2 - x^2}\right]$

(7) For the cycloid $x = a(t + \sin t)$, $y = a(1 - \cos t)$

prove that $\rho = 4a\cos\left(\frac{1}{2}t\right)$

(8) Prove that the radius of curvature at any point of catenary $y = c\cosh\frac{x}{c}$ varies as the square of the ordinate.

(9) Find the radius of curvature for the curve (4)

$r = a(1-\cos\theta)$

(10) prove that for the ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $\rho = \frac{a^2 b^2}{p^3}$; p, being the perpendicular from the centre on the tangent at the point $(x, y)$

(11) If $\rho_1, \rho_2$ be the radii of curvature at the extremities of two conjugate diameters on an ellipse prove that

$$\left(\rho_1^{2/3} + \rho_2^{2/3}\right) a^{2/3} b^{2/3} = a^2 + b^2$$

(12) Find $\rho$ at the origin of the curves

(1) $y = x^4 - 4x^3 - 18x^2$ (2) $3x^3 + y^3 + 5y^2 + 3yx^2 + 2x = 0$

(13) show that evolute of the ellipse $x = a\cos\theta$, $y = b\sin\theta$ is $(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$.

(14) obtain the evolute of the parabola $y^2 = 4ax$.

(15) Find the evolute of the astroid

$x = a\cos^3\theta$, $y = a\sin^3\theta$.

# Unit - III



① If $u = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$ ; $x^2+y^2+z^2 \neq 0$, show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

② If $u = \log(x^3+y^3+z^3-3xyz)$ show that

a) $\left(\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y} + \dfrac{\partial}{\partial z}\right)^2 u = \dfrac{9}{(x+y+z)^2}$

b) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = \dfrac{-3}{(x+y+z)^2}$

③ If $u = f(r)$ where $r = \sqrt{x^2+y^2}$ prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$$

④ If $u = \log(x^2+y^2+z^2)$ prove that

$$x \frac{\partial^2 u}{\partial y \partial z} = y \frac{\partial^2 u}{\partial z \partial x} = z \frac{\partial^2 u}{\partial x \partial y}$$

⑤ If $u = \begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$ show that $u_x + u_y + u_z = 0$

⑥ If $u = \sin^{-1}\dfrac{x}{y} + \tan^{-1}\dfrac{y}{x}$ show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$

⑦ State and prove Euler's theorem on Homogeneous functions

(8) If $u = \tan^{-1} \frac{x^3+y^3}{x+y}$, $x \neq y$ Show that

i) $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \sin 2u$

ii) $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = (1-4\sin^2 u)\sin 2u$

(9) If $u = \log\left\{\frac{x^4+y^4}{x+y}\right\}$ show by Euler's theorem that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 3$

(10) Verify Euler's theorem for

$Z = ax^2 + 2hxy + by^2$

(11) If $u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$, show that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \tan u$

(12) If $u = \tan^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, show that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{1}{4}\sin 2u$

(13) If $z = xy\, f\left(\frac{y}{x}\right)$, show that $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 2z$

~~Unit~~ – I

(14) If $H = f(y-z,\ z-x,\ x-y)$ prove that $\frac{\partial H}{\partial x} + \frac{\partial H}{\partial y} + \frac{\partial H}{\partial z} = 0$

(15) If $x\sqrt{1-y^2} + y\sqrt{1-x^2} = a$ show that $\frac{d^2y}{dx^2} = \frac{a}{(1-x^2)^{3/2}}$

(16) If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ prove that

$$\frac{d^2y}{dx^2} = \frac{abc + 2fgh - af^2 - bg^2 - ch^2}{(hx+by+f)^3}$$

(17) Expand $f(x,y) = x^2 + xy + y^2$ in power of $(x-2)$ and $y-3$.

## unit - IV

(1) find the greatest and least values of the function $2x^3-15x^2+36x+1$ in the interval $[2,3]$

(2) Find the extreme values of $5x^6+18x^5+15x^4-10$

(3) Investigate for maximum and minimum values of the function given by $y = \sin x + \cos 2x$.

(4) Find the maxima and minima of the radii of the curve. $\frac{c^4}{r^2} = \frac{a^2}{\sin^2\theta} + \frac{b^2}{\cos^2\theta}$

(5) Discuss the maximum or minimum values of $u$ given by $u = x^3y^2(1-x-y)$

(6) Show that minimum value of $u = xy + \frac{a^3}{x} + \frac{a^3}{y}$. is $3a^2$

(7) Discuss the maximum and minimum values of

(i) $u = \sin x + \sin y + \sin(x+y)$

(ii) $u = x^3+y^3-3axy$

(8) Find the maxima and minima of $x^2+y^2+z^2$ subject to the conditions $ax^2+by^2+cz^2=1$ and $lx+my+nz=0$

(9) Determine the maxima and minima of (8)

$x^2+y^2+z^2$ when $ax^2+by^2+cz^2=1$

(10) Find the minimum value of $x^2+y^2+z^2$, given that

$ax+by+cz=P$

(11) In a plane triangle find the maximum value

of $u = \cos A \cos B \cos C$

(12) Find the asymptote of $x^3+2x^2y-xy^2-2y^3+xy-y^2-1=0$

(13) Find the asymptote of the cubic curve

$y^3-5xy^2+8x^2y-4x^3-3y^2+9xy-6x^2+2y-2x=1$

(14) Find the asymptote of the curve $y^3-x^2y+2xy^2-y+1=0$

(15) Find the envelope of the family of ellipses

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where the two parameters a and b

are connected by the relation $a+b=c$, c is constant

(16) Assuming that the evolute of a curve is the

envelope of its normals, find the evolute of the

Parabola $y^2=4ax$

(17) Find the envelope of the circles which pass through

the origin and whose centres lies on

a) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ b) $x^2-y^2=a^2$